DEFINING RELATIONS OF FUSION PRODUCTS AND SCHUR POSITIVITY

KATSUYUKI NAOI

Abstract. In this note we give defining relations of an $\mathfrak{sl}_{n+1}[t]$-module defined by the fusion product of simple $\mathfrak{sl}_{n+1}$-modules whose highest weights are multiples of a given fundamental weight. From this result we obtain a surjective homomorphism between two fusion products, which can be considered as a current algebra analog of Schur positivity.

1. Introduction

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ with index set $I = \{1, \dots, n\}$, and fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Denote by ϖ_i $(i \in I)$ the fundamental weights. Let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be the associated current algebra. For $m \in I$ and a sequence $\boldsymbol{\ell} = (\ell_1, \ell_2, \dots, \ell_p)$ of nonnegative integers, we define a $\mathfrak{g}[t]$-module $V_m(\boldsymbol{\ell})$ by

$$V_m(\boldsymbol{\ell}) = V(\ell_1\varpi_m) * V(\ell_2\varpi_m) * \cdots * V(\ell_p\varpi_m).$$

Here $V(\lambda)$ is the simple $\mathfrak{g}$-module with highest weight λ, and $*$ denotes the fusion product defined by Feigin and Loktev in [FL99]. We may assume without loss of generality that $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_p$, that is, $\boldsymbol{\ell}$ is a partition.

In [CV15], Chari and Venkatesh have introduced a large family of indecomposable $\mathfrak{g}[t]$-modules (with $\mathfrak{g}$ a general simple Lie algebra) indexed by a sequence of partitions, in terms of generators and relations. In this note, we will show that the fusion product $V_m(\boldsymbol{\ell})$ is isomorphic to a module belonging to their family. More explicitly, we show the following defining relations of $V_m(\boldsymbol{\ell})$.

Theorem. *Let $m \in I$ and $\boldsymbol{\ell} = (\ell_1 \geq \cdots \geq \ell_p)$ be a partition. Set $L_i = \ell_i + \cdots + \ell_{p-1} + \ell_p$ for $1 \leq i \leq p$ and $L_i = 0$ for $i > p$. Then $V_m(\boldsymbol{\ell})$ is isomorphic to the $\mathfrak{g}[t]$-module generated by a vector v with relations*

$$\mathfrak{n}_+[t]v = 0, \quad (h \otimes t^s)v = \delta_{s0}L_1\langle h, \varpi_m\rangle v \text{ for } h \in \mathfrak{h},\ s \in \mathbb{Z}_{\geq 0},$$
$$(f_\alpha \otimes \mathbb{C}[t])v = 0 \text{ for } \alpha \in \Delta_+ \text{ with } \langle h_\alpha, \varpi_m\rangle = 0,$$
$$f_\alpha^{L_1+1}v = 0 \text{ for } \alpha \in \Delta_+ \text{ with } \langle h_\alpha, \varpi_m\rangle = 1,$$
$$(e_\alpha \otimes t)^s f_\alpha^{r+s}v = 0 \text{ for } \alpha \in \Delta_+, r, s \in \mathbb{Z}_{>0} \text{ with } \langle h_\alpha, \varpi_m\rangle = 1,$$
$$r + s \geq 1 + kr + L_{k+1} \text{ for some } k \in \mathbb{Z}_{>0}.$$

Here Δ_+ is the set of positive roots, h_α is the coroot corresponding to α, and e_α and f_α are root vectors corresponding to α and $-\alpha$ respectively.

1991 *Mathematics Subject Classification.* 17B10.

Key words and phrases. current algebra, fusion product, Schur positivity.

This theorem for $\mathfrak{g} = \mathfrak{sl}_2$ has been proved in [FF02] and [CV15]. In the case $p = 2$, this can be found in [Ven15] and [Fou15] (see also [CSVW14]).

Let us introduce a motivation of the theorem. For that we consider the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ for a moment. Let $(\ell_1 \geq \ell_2), (r_1 \geq r_2)$ be partitions of a positive integer ℓ. By the well-known Clebsch-Gordan formula

$$V(\ell\varpi_1) \otimes V(r\varpi_1) = V\big(|\ell - r|\varpi_1\big) \oplus \cdots \oplus V\big((\ell + r - 2)\varpi_1\big) \oplus V\big((\ell + r)\varpi_1\big),$$

we see that there exists a surjective $\mathfrak{g}$-module homomorphism

$$V(\ell_1\varpi_1) \otimes V(\ell_2\varpi_1) \twoheadrightarrow V(r_1\varpi_1) \otimes V(r_2\varpi_1)$$

if and only if $\ell_2 \geq r_2$. This surjection implies that the difference of their characters can be written as a sum of characters of simple $\mathfrak{g}$-modules. Since the characters of simple $\mathfrak{g}$-modules are known as Schur functions, this property is called *Schur positivity*. Generalization of the surjection to a more general $\mathfrak{g}$ and more general $\mathfrak{g}$-modules has been studied in [DP07, LPP07, CFS14, FH14]. In particular when $\mathfrak{g} = \mathfrak{sl}_{n+1}$, it follows from [CFS14] (see also [LPP07]) that for $m \in I$ and two partitions $(\ell_1 \geq \cdots \geq \ell_p)$, $(r_1 \geq \cdots \geq r_p)$ of a positive integer ℓ, there exists a surjective $\mathfrak{g}$-module homomorphism

$$V(\ell_1\varpi_m) \otimes \cdots \otimes V(\ell_p\varpi_m) \twoheadrightarrow V(r_1\varpi_m) \otimes \cdots \otimes V(r_p\varpi_m) \tag{1.1}$$

if $\ell_i + \cdots + \ell_p \geq r_i + \cdots + r_p$ holds for each $1 \leq i \leq p$.

Fourier and Hernandez have raised the following question in the introduction of [FH14]: Can surjections such as (1.1) be obtained from surjective $\mathfrak{g}[t]$-module homomorphisms between the corresponding fusion products? (Recall that the $\mathfrak{g}$-module structures of a tensor product and a fusion product are the same.) By inspecting the defining relations of the theorem we obtain the following corollary, which gives a positive answer to their question in our setting.

Corollary 1.1. *Let $m \in I$, and $\boldsymbol{\ell} = (\ell_1 \geq \cdots \geq \ell_p)$, $\boldsymbol{r} = (r_1 \geq \cdots \geq r_p)$ be two partitions of a positive integer ℓ. We assume that $\ell_i + \cdots + \ell_p \geq r_i + \cdots + r_p$ holds for each $1 \leq i \leq p$. Then there exists a surjective $\mathfrak{g}[t]$-module homomorphism from $V_m(\boldsymbol{\ell})$ onto $V_m(\boldsymbol{r})$.*

It would be an interesting problem to generalize the theorem to a more general $\mathfrak{g}$ or more general modules. These will be studied elsewhere.

The organization of this paper is as follows. We fix basic notations in Subsection 2.1, and recall the definition of fusion products in Subsection 2.2. By [Nao12] $V_m(\boldsymbol{\ell})$ can be realized as a $\mathfrak{g}[t]$-submodule of a module over the affine Lie algebra $\widehat{\mathfrak{g}}$, which is recalled in Subsection 2.3. In Subsection 2.4, we recall some results in [CV15] needed for the proof of the main theorem, and show one technical lemma. Then we prove the theorem in Section 3 by determining the defining relations recursively using the realization, in which we apply the method used in [Nao13].

2. Preliminaries

2.1. **Simple Lie algebra, current algebra, and affine Kac-Moody Lie algebra of type $\boldsymbol{A}$.** Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ with index set $I = \{1, \ldots, n\}$. We fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Let α_i and ϖ_i $(i \in I)$ be simple roots and fundamental weights respectively. We use the labeling in [Kac90, Section 4.8]. For convenience we set $\varpi_0 = 0$. Let Δ be the root system, Δ_+ the set of positive roots, W the Weyl group with simple reflections

$\{s_i \mid i \in I\}$ and longest element w_0, and $(\ ,\ )$ the unique non-degenerate W-invariant symmetric bilinear form on $\mathfrak{h}^*$ satisfying $(\alpha, \alpha) = 2$ for all $\alpha \in \Delta$. Let

$$\theta = \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n$$

be the highest root. For each $\alpha \in \Delta$, let h_α be its coroot, $\mathfrak{g}_\alpha$ the corresponding root space, and $e_\alpha \in \mathfrak{g}_\alpha$ a root vector satisfying $[e_\alpha, e_{-\alpha}] = h_\alpha$. We also use the notations $f_\alpha = e_{-\alpha}$ for $\alpha \in \Delta_+$, $h_i = h_{\alpha_i}$, $e_i = e_{\alpha_i}$ and $f_i = f_{\alpha_i}$. Denote by P the weight lattice, by P_+ the set of dominant integral weights, and by $V(\lambda)$ ($\lambda \in P_+$) the simple $\mathfrak{g}$-module with highest weight λ. For $i \in I$, set

$$i^* = n + 1 - i \in I.$$

Note that $w_0(\varpi_i) = -\varpi_{i^*}$ holds.

Given a Lie algebra $\mathfrak{a}$, its *current algebra* $\mathfrak{a}[t]$ is defined by the tensor product $\mathfrak{a} \otimes \mathbb{C}[t]$ equipped with the Lie algebra structure given by

$$[x \otimes f(t), y \otimes g(t)] = [x, y] \otimes f(t)g(t).$$

For $k \in \mathbb{Z}_{>0}$, let $t^k\mathfrak{a}[t]$ denote the ideal $\mathfrak{a} \otimes t^k\mathbb{C}[t] \subseteq \mathfrak{a} \otimes \mathbb{C}[t]$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the nontwisted affine Lie algebra associated with $\mathfrak{g}$. Here c is the canonical central element and d is the degree operator. Note that $\mathfrak{g}$ and $\mathfrak{g}[t]$ are naturally considered as Lie subalgebras of $\widehat{\mathfrak{g}}$. Let $\widehat{I} = I \sqcup \{0\}$, and define Lie subalgebras $\widehat{\mathfrak{h}}$, $\widehat{\mathfrak{n}}_+$, and $\widehat{\mathfrak{b}}$ as follows:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \widehat{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus t\mathfrak{g}[t], \quad \widehat{\mathfrak{b}} = \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+.$$

We also define $\widehat{\mathfrak{h}}_{\mathrm{cl}} = \mathfrak{h} \oplus \mathbb{C}c$. We often consider $\mathfrak{h}^*$ (resp. $\widehat{\mathfrak{h}}_{\mathrm{cl}}^*$) as a subspace of $\widehat{\mathfrak{h}}^*$ by setting

$$\langle c, \lambda \rangle = \langle d, \lambda \rangle = 0 \text{ for } \lambda \in \mathfrak{h}^* \quad \big(\text{resp. } \langle d, \lambda \rangle = 0 \text{ for } \lambda \in \widehat{\mathfrak{h}}_{\mathrm{cl}}^*\big).$$

Let $\widehat{\Delta}$ be the root system of $\widehat{\mathfrak{g}}$, $\widehat{P}$ the weight lattice, $\widehat{P}_+$ the set of dominant integral weights, and $\widehat{W}$ the Weyl group with simple reflections $\{s_i \mid i \in \widehat{I}\}$. Denote by $\delta \in \widehat{P}$ the null root, and by $\Lambda_0 \in \widehat{P}$ the unique element satisfying

$$\langle \mathfrak{h}, \Lambda_0 \rangle = \langle d, \Lambda_0 \rangle = 0, \quad \langle c, \Lambda_0 \rangle = 1.$$

Let $\alpha_0 = \delta - \theta$, $e_0 = f_\theta \otimes t$ and $f_0 = e_\theta \otimes t^{-1}$. Given an integrable $\widehat{\mathfrak{g}}$-module M and $i \in \widehat{I}$, define a linear automorphism Φ_i^M on M by

$$\Phi_i^M = \exp(f_i)\exp(-e_i)\exp(f_i),$$

see [Kac90, Lemma 3.8]. For each $w \in \widehat{W}$ fix a reduced expression $w = s_{i_k} \cdots s_{i_1}$, and set $\Phi_w^M = \Phi_{i_k}^M \cdots \Phi_{i_1}^M$. Then Φ_w^M satisfies

$$\Phi_w^M(M_\mu) = M_{w(\mu)} \text{ for } \mu \in \widehat{P}, \quad \Phi_w^M \widehat{\mathfrak{g}}_\alpha (\Phi_w^M)^{-1} = \widehat{\mathfrak{g}}_{w(\alpha)} \text{ for } \alpha \in \widehat{\Delta}.$$

In particular by considering the adjoint representation, an algebra automorphism on $U(\widehat{\mathfrak{g}})$ is defined for each $w \in \widehat{W}$, which is denoted by Φ_w. Note that Φ_w^M for $w \in W$ is also defined on a finite-dimensional $\mathfrak{g}$-module M.

Define $t_\lambda \in \mathrm{GL}(\widehat{\mathfrak{h}}^*)$ for $\lambda \in P$ by

$$t_\lambda(\mu) = \mu + \langle c, \mu \rangle \lambda - \big((\mu, \lambda) + \frac{1}{2}(\lambda, \lambda)\langle c, \mu \rangle\big)\delta,$$

see [Kac90, Chapter 6]. Let $T(P) = \{t_\lambda \mid \lambda \in P\}$ and $\widetilde{W} = W \ltimes T(P)$, which is called the *extended affine Weyl group*. Here $w \in W$ and $t_\lambda \in T(P)$ satisfy $wt_\lambda w^{-1} = t_{w(\lambda)}$. For $i \in \widehat{I}$, let

$$\tau_i = t_{\varpi_i} w_{i,0} w_0 \in \widetilde{W}$$

where $w_{i,0}$ is the longest element of W_{ϖ_i}, the stabilizer of ϖ_i in W. We have

$$\tau_i(\delta) = \delta, \quad \tau_i(\alpha_j) = \alpha_{\overline{i+j}}, \quad \text{and} \quad \tau_i(\varpi_j + \Lambda_0) \equiv \varpi_{\overline{i+j}} + \Lambda_0 \mod \mathbb{Q}\delta \quad \text{for } j \in \widehat{I} \tag{2.1}$$

where $\overline{i+j} \equiv i+j \mod n+1$. Set $\Sigma = \{\tau_i \mid i \in \widehat{I}\}$. It is known that $\widetilde{W} = \widehat{W} \rtimes \Sigma$. We define an action of Σ on $\widehat{\mathfrak{g}}$ by letting τ_i act as a Lie algebra automorphism determined by

$$\tau_i(e_j) = e_{\overline{i+j}}, \quad \tau_i(f_j) = f_{\overline{i+j}} \text{ for } j \in \widehat{I}, \quad \langle \tau_i(h), \tau_i(\lambda) \rangle = \langle h, \lambda \rangle \text{ for } h \in \widehat{\mathfrak{h}}, \lambda \in \widehat{\mathfrak{h}}^*.$$

2.2. **Fusion product.** Let us recall the definition of the fusion product introduced in [FL99]. Note that $U(\mathfrak{g}[t])$ has a natural $\mathbb{Z}_{\geq 0}$-grading defined by

$$U(\mathfrak{g}[t])^k = \{X \in U(\mathfrak{g}[t]) \mid [d, X] = kX\}.$$

Let $\lambda_1, \dots, \lambda_p$ be a sequence of elements of P_+, and $c_1, \dots, c_p$ pairwise distinct complex numbers. We define a $\mathfrak{g}[t]$-module structure on $V(\lambda_i)$ as follows:

$$\big(x \otimes f(t)\big)v = f(c_i)xv \quad \text{for } x \in \mathfrak{g}, f(t) \in \mathbb{C}[t], v \in V(\lambda_i).$$

Denote this $\mathfrak{g}[t]$-module by $V(\lambda_i)_{c_i}$. Let v_i be a highest weight vector of $V(\lambda_i)$. Then the $\mathfrak{g}[t]$-module $V(\lambda_1)_{c_1} \otimes \cdots \otimes V(\lambda_p)_{c_p}$ is generated by $v_1 \otimes \cdots \otimes v_p$ (see [FL99]), and the grading on $U(\mathfrak{g}[t])$ induces a filtration on $V(\lambda_1)_{c_1} \otimes \cdots \otimes V(\lambda_p)_{c_p}$ by

$$\Big(V(\lambda_1)_{c_1} \otimes \cdots \otimes V(\lambda_p)_{c_p}\Big)^{\leq k} = \sum_{r \leq k} U(\mathfrak{g}[t])^r (v_1 \otimes \cdots \otimes v_p).$$

Now the $\mathfrak{g}[t]$-module obtained by taking the associated graded is denoted by

$$V(\lambda_1) * \cdots * V(\lambda_p),$$

and called the *fusion product* of $V(\lambda_1), \dots, V(\lambda_p)$. Though the definition depends on the parameters c_i, we omit them for the notational convenience. All fusion products appearing in this paper do not depend on the parameters up to isomorphism. Note that, by definition, we have

$$V(\lambda_1) * \cdots * V(\lambda_p) \cong V(\lambda_1) \otimes \cdots \otimes V(\lambda_p)$$

as a $\mathfrak{g}$-module.

2.3. **Another realization of fusion products.** Kirillov-Reshetikhin modules for $\mathfrak{g}[t]$ are $\mathfrak{g}[t]$-modules defined in terms of generators and relations, which have been introduced in [CM06]. In [Nao12] the fusion products of Kirillov-Reshetikhin modules for $\mathfrak{g}[t]$ were studied when $\mathfrak{g}$ is of type ADE, and a new realization of these modules using Joseph functors was given. When $\mathfrak{g}$ is of type A, a Kirillov-Reshetikhin module is just the evaluation module at $t = 0$ of $V(k\varpi_i)$ with $k \in \mathbb{Z}_{>0}$ and $i \in I$, and hence their fusion products are what we are studying in this note. In this subsection we will reformulate the result of [Nao12] in type A in a different way (see Remark 2.2). This formulation has previously been used in [Nao13], and is more suitable for later use since we can apply Lemma 2.3 stated below.

First we introduce several notations. Assume that V is a $\widehat{\mathfrak{g}}$-module and D is a $\widehat{\mathfrak{b}}$-submodule of V. For $\tau \in \Sigma$, denote by $F_\tau V$ the pull-back $(\tau^{-1})^* V$ with respect to the Lie algebra automorphism τ^{-1} on $\widehat{\mathfrak{g}}$, and define a $\widehat{\mathfrak{b}}$-submodule $F_\tau D \subseteq F_\tau V$ in the obvious way. For $i \in \widehat{I}$ let $\widehat{\mathfrak{p}}_i$ denote the parabolic subalgebra $\widehat{\mathfrak{b}} \oplus \mathbb{C} f_i \subseteq \widehat{\mathfrak{g}}$, and set $F_i D = U(\widehat{\mathfrak{p}}_i) D \subseteq V$ to be the $\widehat{\mathfrak{p}}_i$-submodule generated by D. Finally we define $F_w D$ for $w \in \widetilde{W}$ as follows: let $\tau \in \Sigma$ and $w' \in \widehat{W}$ be the elements such that $w = w'\tau$, and choose a reduced expression $w' = s_{i_k} \cdots s_{i_1}$. Then we set

$$F_w D = F_{i_k} \cdots F_{i_1} F_\tau D \subseteq F_\tau V.$$

Though the definition depends on the choice of the expression of w', we use F_w by an abuse of notation.

For $\Lambda \in \widehat{P}_+$ let $\widehat{V}(\Lambda)$ be the simple highest weight $\widehat{\mathfrak{g}}$-module with highest weight Λ. Denote by $\mathbb{C}_\Lambda$ the 1-dimensional $\widehat{\mathfrak{b}}$-submodule of $\widehat{V}(\Lambda)$ spanned by a highest weight vector. Note that $F_\tau \widehat{V}(\Lambda) \cong \widehat{V}(\tau\Lambda)$ and $F_\tau \mathbb{C}_\Lambda \cong \mathbb{C}_{\tau\Lambda}$ for $\tau \in \Sigma$. Let

$$\widehat{\mathfrak{b}}' = \widehat{\mathfrak{b}} \cap \mathfrak{g}[t] = \mathfrak{h} \oplus \widehat{\mathfrak{n}}_+.$$

Now [Nao12, Theorem 6.1] is reformulated as follows.

Theorem 2.1. *Let $\boldsymbol{\ell} = (\ell_1 \geq \cdots \geq \ell_p)$ be a partition, and $m_1, \ldots, m_p$ a sequence of elements of I. As a $\widehat{\mathfrak{b}}'$-module, we have*

$$\begin{aligned} &V(\ell_1 \varpi_{m_1}) * \cdots * V(\ell_p \varpi_{m_p}) \\ &\qquad \cong F_{t_{-\varpi_{m_1^*}}} \Big(\mathbb{C}_{(\ell_1 - \ell_2)\Lambda_0} \otimes \cdots \otimes F_{t_{-\varpi_{m_{p-1}^*}}} \big(\mathbb{C}_{(\ell_{p-1} - \ell_p)\Lambda_0} \otimes F_{t_{-\varpi_{m_p^*}}} \mathbb{C}_{\ell_p \Lambda_0} \big) \cdots \Big). \end{aligned}$$

Remark 2.2. In [Nao12, Theorem 6.1] the right-hand side is defined in terms of Joseph functors, but it can easily be proved to be isomorphic to the right-hand side of Theorem 2.1 as follows. By the universality of Joseph functors, there exists a surjection between two modules. Moreover their characters coincide by [Nao12, Corollary 6.2] and [LLM02, Theorem 5], and hence they are isomorphic. (See [Nao13, a paragraph below Lemma 5.2] for more detail, in which a similar argument is given.)

For $i \in \widehat{I}$, let $\widehat{\mathfrak{n}}_i$ be the nilradical of $\widehat{\mathfrak{p}}_i$. More explicitly, $\widehat{\mathfrak{n}}_i$ is defined as follows:

$$\widehat{\mathfrak{n}}_i = \bigoplus_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} \mathbb{C} e_\alpha \oplus t\mathfrak{g}[t] \ (i \in I), \ \widehat{\mathfrak{n}}_0 = \mathfrak{n}_+ \oplus \bigoplus_{\alpha \in \Delta \setminus \{-\theta\}} \mathbb{C}(e_\alpha \otimes t) \oplus (\mathfrak{h} \otimes t) \oplus t^2 \mathfrak{g}[t].$$

The following lemma is useful to determine defining relations of modules constructed using F_w's. For the proof, see [Nao13, Lemma 5.3].

Lemma 2.3. *Let V be an integrable $\widehat{\mathfrak{g}}$-module, T a finite-dimensional $\widehat{\mathfrak{b}}$-submodule of V, $i \in \widehat{I}$ and $\xi \in \widehat{P}$ such that $\langle h_i, \xi \rangle \geq 0$. Assume that the following conditions hold:*
(i) *T is generated by a weight vector $v \in T_\xi$ satisfying $e_i v = 0$.*
(ii) *There is an $ad(e_i)$-invariant left $U(\widehat{\mathfrak{n}}_i)$-ideal $\mathcal{I}$ such that*

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v = U(\widehat{\mathfrak{n}}_+) e_i + U(\widehat{\mathfrak{n}}_+) \mathcal{I}.$$

(iii) *We have* $\mathrm{ch}\, F_i T = \mathcal{D}_i \mathrm{ch}\, T$, *where* ch *denotes the character with respect to* $\widehat{\mathfrak{h}}$, *and* $\mathcal{D}_i$ *is the Demazure operator defined by*

$$\mathcal{D}_i(f) = \frac{f - e^{-\alpha_i} s_i(f)}{1 - e^{-\alpha_i}}.$$

Let $v' = f_i^{\langle h_i, \xi\rangle} v$. *Then we have*

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v' = U(\widehat{\mathfrak{n}}_+) e_i^{\langle h_i,\xi\rangle + 1} + U(\widehat{\mathfrak{n}}_+)\Phi_i(\mathcal{I}).$$

2.4. **Presentation by Chari and Venkatesh.** Following [CV15], we introduce some notations. For $r, s \in \mathbb{Z}_{\geq 0}$, let

$$\mathbf{S}(r,s) = \Big\{ (b_j)_{j\geq 0} \;\Big|\; b_j \in \mathbb{Z}_{\geq 0},\ \sum_j b_j = r,\ \sum_j j b_j = s \Big\}.$$

Note that $\mathbf{S}(0,s) = \emptyset$ if $s > 0$, and if $(b_j)_{j\geq 0} \in \mathbf{S}(r,s)$ then $b_j = 0$ for $j > s$. For $x \in \mathfrak{g}$ and $r, s \in \mathbb{Z}_{\geq 0}$, define a vector $x(r,s) \in U(\mathfrak{g}[t])$ by

$$x(r,s) = \sum_{(b_j)_{j\geq 0} \in \mathbf{S}(r,s)} (x\otimes 1)^{(b_0)} (x \otimes t)^{(b_1)} \cdots (x \otimes t^s)^{(b_s)},$$

where for $X \in \mathfrak{g}[t]$, $X^{(b)}$ denotes the divided power $X^b/b!$. We understand $x(r,s) = 0$ if $\mathbf{S}(r,s) = \emptyset$. For $\alpha \in \Delta_+$, define Lie subalgebras $\mathfrak{sl}_{2,\alpha}$ and $\mathfrak{b}_\alpha$ of $\mathfrak{g}$ by

$$\mathfrak{sl}_{2,\alpha} = \mathbb{C}e_\alpha \oplus \mathbb{C}h_\alpha \oplus \mathbb{C}f_\alpha, \quad \mathfrak{b}_\alpha = \mathbb{C}e_\alpha \oplus \mathbb{C}h_\alpha.$$

We also define a Lie subalgebra $\widehat{\mathfrak{m}}_\alpha$ of $\mathfrak{sl}_{2,\alpha}[t]$ by

$$\widehat{\mathfrak{m}}_\alpha = t\mathfrak{sl}_{2,\alpha}[t] \oplus \mathbb{C}f_\alpha.$$

By [Gar78] (see also [CP01, Lemma 1.3]), we have

$$(e_\alpha \otimes t)^{(s)} f_\alpha^{(r+s)} - (-1)^s f_\alpha(r,s) \in U(\widehat{\mathfrak{m}}_\alpha) t \mathfrak{b}_\alpha[t]. \tag{2.2}$$

For $k \in \mathbb{Z}_{\geq 0}$, let $\mathbf{S}(r,s)_k$ (resp. ${}_k\mathbf{S}(r,s)$) be the subset of $\mathbf{S}(r,s)$ consisting of elements $(b_j)_{j\geq 0}$, satisfying

$$b_j = 0 \ \text{ for } \ j \geq k \quad (\text{resp. } b_j = 0 \ \text{ for } \ j < k).$$

For $x \in \mathfrak{g}$, define a vector $x(r,s)_k$ and ${}_kx(r,s)$ by

$$x(r,s)_k = \sum_{(b_j)_{j\geq 0} \in \mathbf{S}(r,s)_k} (x\otimes 1)^{(b_0)} (x\otimes t)^{(b_1)} \cdots (x \otimes t^{k-1})^{(b_{k-1})},$$

$${}_kx(r,s) = \sum_{(b_j)_{j\geq 0} \in {}_k\mathbf{S}(r,s)} (x\otimes t^k)^{(b_k)} (x \otimes t^{k+1})^{(b_{k+1})} \cdots (x\otimes t^s)^{(b_s)}.$$

The following was proved in [CV15].

Lemma 2.4. (i) *Let* $x \in \mathfrak{g}$. *If* $r, s, k \in \mathbb{Z}_{>0}$ *and* $K \in \mathbb{Z}_{\geq 0}$ *satisfy* $r + s \geq kr + K$, *then*

$$x(r,s) = {}_kx(r,s) + \sum_{(r',s')} x(r-r', s-s')_k \cdot {}_kx(r',s'),$$

where the sum is over all pairs $r', s' \in \mathbb{Z}_{\geq 0}$ *such that* $r' < r, s' \leq s$ *and* $r' + s' \geq kr' + K$.
(ii) *Let* $\alpha \in \Delta_+$, V *be an* $\mathfrak{sl}_{2,\alpha}[t]$*-module,* $v \in V$ *and* $K \in \mathbb{Z}_{\geq 0}$. *Assume that* $e_\alpha \otimes \mathbb{C}[t]$ *and* $h_\alpha \otimes t\mathbb{C}[t]$ *act trivially on* $v \in V$. *Then,*

$$(e_\alpha \otimes t)^s f_\alpha^{r+s} v = 0 \text{ for all } r, s \in \mathbb{Z}_{>0} \text{ with } r + s \geq 1 + kr + K \text{ for some } k \in \mathbb{Z}_{>0}$$

if and only if

$$_k f_\alpha(r, s) v = 0 \quad \text{for all } r, s, k \in \mathbb{Z}_{>0} \text{ with } r + s \geq 1 + kr + K.$$

The following proposition plays an important roll in the next section.

Proposition 2.5. *Let* $\alpha \in \Delta_+$. *If* $r, s, k \in \mathbb{Z}_{>0}$ *and* $K \in \mathbb{Z}_{\geq 0}$ *satisfy* $r + s \geq kr + K$, *then we have*

$$\big[e_\alpha,\ {}_k f_\alpha(r, s)\big] \in \sum_{(r',s')} U(t\mathfrak{sl}_{2,\alpha}[t])_k f_\alpha(r', s') + U(t\mathfrak{sl}_{2,\alpha}[t]) t\mathfrak{b}_\alpha[t],$$

where the sum is over all pairs $r', s' \in \mathbb{Z}_{>0}$ *such that* $r' + s' \geq kr' + K$.

Proof. First we introduce some notation. We write $\mathfrak{f}_\alpha = \mathbb{C} f_\alpha$ here. Define Lie subalgebras $\widehat{\mathfrak{m}}_\alpha^h$, $\mathfrak{f}_\alpha[t]_{<k}$ and $\mathfrak{f}_\alpha[t]_{<k}^h$ by

$$\widehat{\mathfrak{m}}_\alpha^h = \widehat{\mathfrak{m}}_\alpha \oplus \mathbb{C} h_\alpha, \quad \mathfrak{f}_\alpha[t]_{<k} = \bigoplus_{j=0}^{k-1} \mathbb{C}(f_\alpha \otimes t^j), \quad \mathfrak{f}_\alpha[t]_{<k}^h = \mathfrak{f}_\alpha[t]_{<k} \oplus \mathbb{C} h_\alpha.$$

Since

$$\widehat{\mathfrak{m}}_\alpha^h = \mathfrak{f}_\alpha[t]_{<k}^h \oplus t^k \mathfrak{f}_\alpha[t] \oplus t\mathfrak{b}_\alpha[t],$$

we have by the PBW theorem that

$$U(\widehat{\mathfrak{m}}_\alpha^h) = \mathfrak{f}_\alpha[t]_{<k}^h U(\widehat{\mathfrak{m}}_\alpha^h) \oplus U\big(t^k \mathfrak{f}_\alpha[t] \oplus t\mathfrak{b}_\alpha[t]\big).$$

Denote by p the projection

$$U(\widehat{\mathfrak{m}}_\alpha^h) \twoheadrightarrow U\big(t^k \mathfrak{f}_\alpha[t] \oplus t\mathfrak{b}_\alpha[t]\big)$$

with respect to this decomposition. It follows from Lemma 2.4 (i) that

$$p\big(f_\alpha(r', s')\big) = {}_k f_\alpha(r', s'). \tag{2.3}$$

Denote by $\mathcal{I}$ the left $U(t\mathfrak{sl}_{2,\alpha}[t])$-ideal in the assertion.

Now we begin the proof of the proposition. By (2.2), it follows that

$$\big[e_\alpha,\ (e_\alpha \otimes t)^{(s)} f_\alpha^{(r+s)}\big] - (-1)^s \big[e_\alpha,\ f_\alpha(r, s)\big] \in U(\widehat{\mathfrak{m}}_\alpha^h) t\mathfrak{b}_\alpha[t].$$

By applying p to this, we have

$$p\Big(\big[e_\alpha,\ (e_\alpha \otimes t)^{(s)} f_\alpha^{(r+s)}\big]\Big) - (-1)^s p\Big(\big[e_\alpha,\ f_\alpha(r, s)\big]\Big) \in U\big(t^k \mathfrak{f}_\alpha[t] \oplus t\mathfrak{b}_\alpha[t]\big) t\mathfrak{b}_\alpha[t] \subseteq \mathcal{I}. \tag{2.4}$$

The following calculation is elementary:

$$\big[e_\alpha,\ (e_\alpha \otimes t)^{(s)} f_\alpha^{(r+s)}\big] = (e_\alpha \otimes t)^{(s)} \big[e_\alpha,\ f_\alpha^{(r+s)}\big] = (h_\alpha + r - s - 1)(e_\alpha \otimes t)^{(s)} f_\alpha^{(r+s-1)}.$$

Note that the pair $(r-1,s)$ satisfies the condition $(r-1)+s \geq k(r-1)+K$ since $k \in \mathbb{Z}_{>0}$. By (2.2), the above equality implies

$$\begin{aligned} p\Big(\big[e_\alpha,\ (e_\alpha \otimes t)^{(s)} f_\alpha^{(r+s)}\big]\Big) &\in p\Big(\mathbb{C}(e_\alpha \otimes t)^{(s)} f_\alpha^{(r+s-1)}\Big) \\ &\subseteq p\Big(\mathbb{C} f_\alpha(r-1,s) + U(\widehat{\mathfrak{m}}_\alpha) t\mathfrak{b}_\alpha[t]\Big) \\ &= \mathbb{C}\,{}_k f_\alpha(r-1,s) + U\big(t^k \mathfrak{f}_\alpha[t] \oplus t\mathfrak{b}_\alpha[t]\big) t\mathfrak{b}_\alpha[t] \subseteq \mathcal{I}, \end{aligned} \tag{2.5}$$

where the equality holds by (2.3). On the other hand, we have by Lemma 2.4 (i) that

$$p\Big(\big[e_\alpha,\ f_\alpha(r,s)\big]\Big) = p\Big(\big[e_\alpha,\ {}_k f_\alpha(r,s)\big]\Big) + \sum_{(r',s')} p\Big(\big[e_\alpha,\ f_\alpha(r-r',s-s')_k \cdot {}_k f_\alpha(r',s')\big]\Big). \tag{2.6}$$

Since $\big[e_\alpha,\ {}_k f_\alpha(r,s)\big] \in U(t^k \mathfrak{sl}_{2,\alpha}[t])$, it follows that $p\Big(\big[e_\alpha,\ {}_k f_\alpha(r,s)\big]\Big) = \big[e_\alpha,\ {}_k f_\alpha(r,s)\big]$, and

$$\begin{aligned} &p\Big(\big[e_\alpha,\ f_\alpha(r-r',s-s')_k \cdot {}_k f_\alpha(r',s')\big]\Big) \\ &\quad = p\Big(\big[e_\alpha,\ f_\alpha(r-r',s-s')_k\big]{}_k f_\alpha(r',s')\Big) + p\Big(f_\alpha(r-r',s-s')_k\big[e_\alpha,\ {}_k f_\alpha(r',s')\big]\Big) \\ &\quad \in p\Big(U(\widehat{\mathfrak{m}}_\alpha^h)_k f_\alpha(r',s')\Big) + 0 = U\big(t^k \mathfrak{f}_\alpha[t] \oplus t\mathfrak{b}_\alpha[t]\big)_k f_\alpha(r',s') \subseteq \mathcal{I}. \end{aligned}$$

Hence (2.6) implies

$$p\Big(\big[e_\alpha,\ f_\alpha(r,s)\big]\Big) - \big[e_\alpha,\ {}_k f_\alpha(r,s)\big] \in \mathcal{I}.$$

Now $\big[e_\alpha,\ {}_k f_\alpha(r,s)\big] \in \mathcal{I}$ follows from this, together with (2.4) and (2.5). The proof is complete. □

3. Main theorem and proof

Let $m \in I$ and $\boldsymbol{\ell} = (\ell_1 \geq \cdots \geq \ell_p)$ be a partition, and denote by $V_m(\boldsymbol{\ell})$ the fusion product $V(\ell_1\varpi_m) * \cdots * V(\ell_p\varpi_m)$. Set $L_i = \ell_i + \cdots + \ell_p$ for $1 \leq i \leq p$, and $L_i = 0$ for $i > p$. As mentioned in the introduction, the main theorem of this note is the following.

Theorem 3.1. *The fusion product $V_m(\boldsymbol{\ell})$ is isomorphic to the $\mathfrak{g}[t]$-module generated by a vector v with relations*

$$\begin{aligned} \mathfrak{n}_+[t]v = 0, \quad (h \otimes t^s)v &= \delta_{s0} L_1 \langle h, \varpi_m\rangle v \text{ for } h \in \mathfrak{h},\ s \in \mathbb{Z}_{\geq 0}, \\ \big(f_\alpha \otimes \mathbb{C}[t]\big)v &= 0 \text{ for } \alpha \in \Delta_+ \text{ with } \langle h_\alpha, \varpi_m\rangle = 0, \\ f_\alpha^{L_1+1} v &= 0 \text{ for } \alpha \in \Delta_+ \text{ with } \langle h_\alpha, \varpi_m\rangle = 1, \\ (e_\alpha \otimes t)^s f_\alpha^{r+s} v &= 0 \text{ for } \alpha \in \Delta_+, r, s \in \mathbb{Z}_{>0} \text{ with } \langle h_\alpha, \varpi_m\rangle = 1, \\ & \qquad r + s \geq 1 + kr + L_{k+1} \text{ for some } k \in \mathbb{Z}_{>0}. \end{aligned}$$

Remark 3.2. In [CV15], the authors have introduced a collection of $\mathfrak{g}[t]$-modules $V(\boldsymbol{\xi})$ (with $\mathfrak{g}$ a general simple Lie algebra) indexed by a $|\Delta_+|$-tuple of partitions $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha \in \Delta_+}$

satisfying $|\xi^\alpha| = \langle h_\alpha, \lambda\rangle$ for some $\lambda \in P_+$. In their terminology, the theorem says that $V_m(\boldsymbol{\ell})$ is isomorphic to $V(\boldsymbol{\xi})$ where $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha\in\Delta_+}$ with

$$\xi^\alpha = \begin{cases} \boldsymbol{\ell} & \text{if } \langle h_\alpha, \varpi_m\rangle = 1, \\ 0 & \text{if } \langle h_\alpha, \varpi_m\rangle = 0. \end{cases}$$

The proof of the theorem will occupy the rest of this paper. Fix $m \in I$ and $\boldsymbol{\ell}$ from now on. By Theorem 2.1, we have

$$V_m(\boldsymbol{\ell}) \cong F_{t_{-\varpi_{m^*}}}\Big(\mathbb{C}_{(\ell_1-\ell_2)\Lambda_0} \otimes \cdots \otimes F_{t_{-\varpi_{m^*}}}\big(\mathbb{C}_{(\ell_{p-1}-\ell_p)\Lambda_0} \otimes F_{t_{-\varpi_{m^*}}}\mathbb{C}_{\ell_p\Lambda_0}\big)\cdots\Big) \tag{3.1}$$

as $\widehat{\mathfrak{b}}'$-modules. We shall determine defining relations of the right-hand side recursively. In the sequel, we write $\tau = \tau_m$ and $\sigma = w_0 w_{m,0}$ (see Subsection 2.1). Note that

$$\sigma(\varpi_m) = w_0(\varpi_m) = -\varpi_{m^*} \quad \text{and} \quad t_{-\varpi_{m^*}} = \sigma t_{\varpi_m}\sigma^{-1} = \sigma\tau$$

hold. Let $\sigma = s_{i_{\ell(\sigma)}} \cdots s_{i_2}s_{i_1}$ be a reduced expression of σ, and set $\sigma_j = s_{i_j}\cdots s_{i_2}s_{i_1}$ for $0 \le j \le \ell(\sigma)$. For $a \in \{0, \pm 1\}$, define a subset $\Delta[a] \subseteq \Delta$ by

$$\Delta[a] = \{\alpha \in \Delta \mid \langle h_\alpha, \varpi_m\rangle = a\}.$$

Note that $\Delta[\pm 1] \subseteq \pm\Delta_+$, and

$$\alpha \in \Delta[a] \text{ if and only if } \langle \sigma(h_\alpha), \varpi_{m^*}\rangle = -a. \tag{3.2}$$

We also write $\Delta[\geq 0] = \Delta[0] \sqcup \Delta[1]$, etc. It should be noted that, since σ is the shortest element such that $\sigma(\varpi_m) = -\varpi_{m^*}$, for every $1 \le j \le \ell(\sigma)$ we have

$$\langle h_{i_j}, \sigma_{j-1}(\varpi_m)\rangle = 1 \text{ and } \sigma_{j-1}^{-1}(\alpha_{i_j}) \in \Delta[1]. \tag{3.3}$$

Define a parabolic subalgebra $\mathfrak{p}_{\varpi_m}$ of $\mathfrak{g}$ by

$$\mathfrak{p}_{\varpi_m} = \bigoplus_{\alpha\in\Delta[\geq 0]} \mathbb{C}e_\alpha \oplus \mathfrak{h} = \bigoplus_{\alpha\in\Delta[0]\cap\Delta_+} \mathbb{C}f_\alpha \oplus \mathfrak{b}.$$

For $1 \le q \le p$ and $0 \le j \le \ell(\sigma)$, let $V(q,j)$ be the $\widehat{\mathfrak{b}}$-module

$$F_{\sigma_j\tau}\Big(\mathbb{C}_{(\ell_q-\ell_{q+1})\Lambda_0} \otimes F_{t_{-\varpi_{m^*}}}\Big(\cdots \otimes F_{t_{-\varpi_{m^*}}}\big(\mathbb{C}_{(\ell_{p-1}-\ell_p)\Lambda_0} \otimes F_{t_{-\varpi_{m^*}}}\mathbb{C}_{\ell_p\Lambda_0}\big)\cdots\Big)\Big).$$

Proposition 3.3. *For every q and j, there exists a nonzero vector $v_{q,j}$ in $V(q,j)$ whose $\widehat{\mathfrak{h}}_{\mathrm{cl}}$-weight is $L_q\sigma_j(\varpi_m) + \ell_q\Lambda_0$, such that $V(q,j)$ is generated by $v_{q,j}$ as a $\widehat{\mathfrak{b}}'$-module and*

$$\begin{aligned}\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v_{q,j} = \sum_{\substack{\alpha\in\Delta[-1] \\ \sigma_j(\alpha)\in\Delta_+}} U(\widehat{\mathfrak{n}}_+)e_{\sigma_j(\alpha)}^{L_q+1} + \sum_{\substack{\alpha\in\Delta[\geq 0] \\ \sigma_j(\alpha)\in\Delta_+}} U(\widehat{\mathfrak{n}}_+)e_{\sigma_j(\alpha)} \\ + \sum_{\alpha\in\Delta[-1]}\sum_{(r,s,k)} U(\widehat{\mathfrak{n}}_+)_k e_{\sigma_j(\alpha)}(r,s) + U(\widehat{\mathfrak{n}}_+)\Phi_{\sigma_j}\big(t\mathfrak{p}_{\varpi_m}[t]\big),\end{aligned} \tag{3.4}$$

where the sum for (r,s,k) is over all $r,s,k \in \mathbb{Z}_{>0}$ such that $r + s \geq 1 + kr + L_{k+q}$.

For a while we assume this proposition, and give a proof to Theorem 3.1. Denote by T_q the running set of (r,s,k) in (3.4), that is,

$$T_q = \big\{(r,s,k) \in \mathbb{Z}^3_{>0} \bigm| r + s \geq 1 + kr + L_{k+q}\big\}.$$

Since $\langle h_\alpha, \varpi_{m^*}\rangle = -\langle h_{\sigma^{-1}(\alpha)}, \varpi_m\rangle$, we see that for $\alpha \in \Delta_+$, $\sigma^{-1}(\alpha) \in \Delta[-1]$ is equivalent to $\langle h_\alpha, \varpi_{m^*}\rangle = 1$. Hence (3.1) and Proposition 3.3 with $q = 1$ and $j = \ell(\sigma)$ imply that there exists a nonzero vector v' in $V_m(\boldsymbol{\ell})$ whose $\mathfrak{h}$-weight is $-L_1\varpi_{m^*}$, such that $V_m(\boldsymbol{\ell})$ is generated by v' and

$$\begin{aligned}\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}v' = \sum_{\alpha\in\Delta_+} U(\widehat{\mathfrak{n}}_+)e_\alpha^{L_1\langle h_\alpha,\varpi_{m^*}\rangle+1} \\ + \sum_{\alpha\in\Delta[-1]}\sum_{(r,s,k)\in T_1} U(\widehat{\mathfrak{n}}_+)_k e_{\sigma(\alpha)}(r,s) + U(\widehat{\mathfrak{n}}_+)\Phi_\sigma(t\mathfrak{p}_{\varpi_m}[t]),\end{aligned} \tag{3.5}$$

where the first summation in the right-hand side is obtained using (3.2). $V_m(\boldsymbol{\ell})$ being a finite-dimensional $\mathfrak{g}$-module, $\Phi_{w_0}^{V_m(\boldsymbol{\ell})}$ is defined. Set $v'' = \Phi_{w_0}^{V_m(\boldsymbol{\ell})}(v') \in V_m(\boldsymbol{\ell})_{L_1\varpi_m}$, and $\widehat{\mathfrak{m}}_+ = \Phi_{w_0}(\widehat{\mathfrak{n}}_+) = \mathfrak{n}_-[t]\oplus t\mathfrak{b}[t]$. Since each $\Delta[a]$ is stable by $w_0\sigma = w_{m,0}$ and $\Delta[-1] = -\Delta[1]$, it follows that

$$\mathrm{Ann}_{U(\widehat{\mathfrak{m}}_+)}v'' = \sum_{\alpha\in\Delta_+} U(\widehat{\mathfrak{m}}_+)f_\alpha^{L_1\langle h_\alpha,\varpi_m\rangle+1} + \sum_{\alpha\in\Delta[1]}\sum_{(r,s,k)\in T_1} U(\widehat{\mathfrak{m}}_+)_k f_\alpha(r,s) + U(\widehat{\mathfrak{m}}_+)t\mathfrak{p}_{\varpi_m}[t].$$

Let M be the $\mathfrak{g}[t]$-module generated by a vector v with relations in Theorem 3.1. By Lemma 2.4 (ii), v satisfies

$${}_kf_\alpha(r,s)v = 0 \text{ for } \alpha\in\Delta[1], (r,s,k)\in T_1.$$

Then we see from the above description of $\mathrm{Ann}_{U(\widehat{\mathfrak{m}}_+)}v''$ that there exists a surjective $\widehat{\mathfrak{m}}_+$-module homomorphism from $V_m(\boldsymbol{\ell})$ to M mapping v'' to v. On the other hand, since

$$V_m(\boldsymbol{\ell}) \cong V(\ell_1\varpi_m)\otimes\cdots\otimes V(\ell_p\varpi_m)$$

as a $\mathfrak{g}$-module, we have $V_m(\boldsymbol{\ell})_\mu = 0$ if $\mu > L_1\varpi_m$, which implies $\mathfrak{n}_+v'' = 0$. Then again by Lemma 2.4 (ii), v'' satisfies $(e_\alpha\otimes t)^s f_\alpha^{r+s}v'' = 0$ for $\alpha \in \Delta[1]$ and r, s with $(r,s,k) \in T_1$ for some $k \in \mathbb{Z}_{>0}$, and we also see that there exists a surjective $\mathfrak{g}[t]$-module homomorphism from M to $V_m(\boldsymbol{\ell})$ mapping v to v''. Hence $V_m(\boldsymbol{\ell}) \cong M$ holds, and the theorem is proved.

The rest of this paper is devoted to prove Proposition 3.3. Define a left $U(\widehat{\mathfrak{n}}_+)$-ideal $\mathcal{I}(q,j)$ by the right-hand side of (3.4). We prove the assertion by the induction on (q,j). When $q = p$ and $j = 0$,

$$V(p,0) = F_\tau\mathbb{C}_{\ell_p\Lambda_0} \cong \mathbb{C}_{\ell_p(\varpi_m+\Lambda_0)}$$

is a 1-dimensional module with $\widehat{\mathfrak{h}}_{\mathrm{cl}}$-weight $\ell_p(\varpi_m+\Lambda_0)$ on which $\widehat{\mathfrak{n}}_+$ acts trivially. Hence in order to verify the assertion in this case, it suffices to show that $\mathcal{I}(p,0) = U(\widehat{\mathfrak{n}}_+)$. The containment $\mathcal{I}(p,0) \subseteq U(\widehat{\mathfrak{n}}_+)$ is obvious, and $\mathfrak{n}_+ + t\mathfrak{p}_{\varpi_m}[t] \subseteq \mathcal{I}(p,0)$ is easily seen. Moreover since $L_{1+p} = 0$, $(1,s,s) \in T_p$ for every $s \in \mathbb{Z}_{>0}$, and hence we have

$${}_se_\alpha(1,s) = e_\alpha\otimes t^s \in \mathcal{I}(p,0) \text{ for } \alpha\in\Delta[-1],\ s\in\mathbb{Z}_{>0}.$$

Hence $U(\widehat{\mathfrak{n}}_+) \subseteq \mathcal{I}(p,0)$ holds.

Next we shall prove that, if the assertion for $(q,j-1)$ holds, then that for (q,j) also holds. We write $i = i_j$ for short. We have $V(q,j) = F_iV(q,j-1)$, and the $\widehat{\mathfrak{h}}_{\mathrm{cl}}$-weight of $v_{q,j-1}$ is $L_q\sigma_{j-1}(\varpi_m) + \ell_q\Lambda_0$. Moreover $e_iv_{q,j-1} = 0$ holds by (3.3). Set $v_{q,j} = f_i^{L_q}v_{q,j-1}$. Since $V(q,j)$ is a submodule of an integrable module, it follows from the representation theory of $\mathfrak{sl}_2$ that

$$v_{q,j}\neq 0,\quad f_iv_{q,j} = 0,\quad\text{and}\quad e_i^{L_q}v_{q,j}\in\mathbb{C}^\times v_{q,j-1}.$$

Hence we have

$$V(q,j) = F_i V(q,j-1) = U(\widehat{\mathfrak{p}}_i)v_{q,j-1} = U(\widehat{\mathfrak{p}}_i)v_{q,j} = U(\widehat{\mathfrak{b}}')v_{q,j},$$

and the cyclicity of $V(q,j)$ is proved. Moreover it is obvious that the $\widehat{\mathfrak{h}}_{\mathrm{cl}}$-weight of $v_{q,j}$ is $L_q\sigma_j(\varpi_m) + \ell_q\Lambda_0$. It remains to prove $\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}(v_{q,j}) = \mathcal{I}(q,j)$. Let $\mathcal{J}$ be the left $U(\widehat{\mathfrak{n}}_i)$-ideal defined by

$$\mathcal{J} = \sum_{\substack{\alpha\in\Delta[-1]\\ \sigma_{j-1}(\alpha)\in\Delta_+}} U(\widehat{\mathfrak{n}}_i)e_{\sigma_{j-1}(\alpha)}^{L_q+1} + \sum_{\substack{\alpha\in\Delta[\geq 0]\\ \sigma_{j-1}(\alpha)\in\Delta_+\setminus\{\alpha_i\}}} U(\widehat{\mathfrak{n}}_i)e_{\sigma_{j-1}(\alpha)}$$
$$+ \sum_{\alpha\in\Delta[-1]}\sum_{(r,s,k)\in T_q} U(\widehat{\mathfrak{n}}_i)_k e_{\sigma_{j-1}(\alpha)}(r,s) + U(\widehat{\mathfrak{n}}_i)\Phi_{\sigma_{j-1}}\big(t\mathfrak{p}_{\varpi_m}[t]\big).$$

By the induction hypothesis we have

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}v_{q,j-1} = \mathcal{I}(q,j-1) = U(\widehat{\mathfrak{n}}_+)e_i + U(\widehat{\mathfrak{n}}_+)\mathcal{J}.$$

It suffices to show that $V(q,j-1)$, $v_{q,j-1}$ and $\mathcal{J}$ satisfy the conditions (i)–(iii) in Lemma 2.3. Indeed if they satisfy the conditions, it follows from the lemma that

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}v_{q,j} = U(\widehat{\mathfrak{n}}_+)e_i^{L_q+1} + U(\widehat{\mathfrak{n}}_+)\Phi_i(\mathcal{J}) = \mathcal{I}(q,j),$$

as required. The condition (i) follows from the induction hypothesis, and the condition (iii) is proved by [LLM02, Theorem 5], or [Nao12, Corollary 2.13 and Lemma 3.2(ii)]. In order to show the condition (ii) we need to prove that $\mathcal{J}$ is $\mathrm{ad}(e_i)$-invariant. For that, we first verify for $\alpha \in \Delta\setminus\{-\alpha_i\}$ that

$$\mathrm{ad}(e_i)U\big(t\mathfrak{g}_\alpha[t]\big) \subseteq \mathcal{J}.$$

If $\alpha+\alpha_i \notin \Delta$, this is obvious. Assume that $\alpha+\alpha_i\in\Delta$. Then it follows from (3.3) that

$$\alpha+\alpha_i = \sigma_{j-1}\big(\sigma_{j-1}^{-1}(\alpha) + \sigma_{j-1}^{-1}(\alpha_i)\big) \in \sigma_{j-1}\big(\Delta[\geq 0]\big).$$

Since $[e_{\alpha+\alpha_i}, e_\alpha] = 0$ holds, this implies

$$\mathrm{ad}(e_i)U\big(t\mathfrak{g}_\alpha[t]\big) \subseteq U(t\mathfrak{g}_\alpha[t])t\mathfrak{g}_{\alpha+\alpha_i}[t] \subseteq \mathcal{J},$$

as required. In a similar manner, $\mathrm{ad}(e_i)U\big(\mathbb{C}e_\alpha\big) \subseteq \mathcal{J}$ for $\alpha\in\Delta_+$ is also proved. In addition, $\mathrm{ad}(e_i)\big(t\mathfrak{h}[t]\big) = t\mathfrak{g}_{\alpha_i}[t] \subseteq \mathcal{J}$ follows from (3.3). Now combining these facts with Proposition 2.5, $\mathrm{ad}(e_i)\big(\mathcal{J}\big)\subseteq\mathcal{J}$ is proved.

Finally it remains to prove that the assertion for $\big(q+1,\ell(\sigma)\big)$ implies that for $(q,0)$. Note that

$$V(q,0) = F_\tau\Big(\mathbb{C}_{(\ell_q-\ell_{q+1})\Lambda_0}\otimes V\big(q+1,\ell(\sigma)\big)\Big) \cong \mathbb{C}_{(\ell_q-\ell_{q+1})(\varpi_m+\Lambda_0)}\otimes(\tau^{-1})^* V\big(q+1,\ell(\sigma)\big).$$

Let z be a basis of $\mathbb{C}_{(\ell_q-\ell_{q+1})(\varpi_m+\Lambda_0)}$ and set $v_{q,0} = z\otimes(\tau^{-1})^*v_{q+1,\ell(\sigma)}$, where $(\tau^{-1})^*v_{q+1,\ell(\sigma)}$ is the image of $v_{q+1,\ell(\sigma)}$ under the linear isomorphism $V\big(q+1,\ell(\sigma)\big)\to(\tau^{-1})^*V\big(q+1,\ell(\sigma)\big)$.

By (2.1), the $\widehat{\mathfrak{h}}_{\mathrm{cl}}$-weight of $v_{q,0}$ is

$$\begin{aligned}(\ell_q-\ell_{q+1})(\varpi_m+\Lambda_0)+\tau(-L_{q+1}\varpi_{m^*}+\ell_{q+1}\Lambda_0)\\ &=(\ell_q-\ell_{q+1})(\varpi_m+\Lambda_0)+\tau\big(-L_{q+1}(\varpi_{m^*}+\Lambda_0)+(\ell_{q+1}+L_{q+1})\Lambda_0\big)\\ &=(\ell_q-\ell_{q+1})(\varpi_m+\Lambda_0)+\big(-L_{q+1}\Lambda_0+(\ell_{q+1}+L_{q+1})(\varpi_m+\Lambda_0)\big)\\ &=L_q\varpi_m+\ell_q\Lambda_0.\end{aligned}$$

Moreover since $\widehat{\mathfrak{n}}_+$ acts trivially on z, we have

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}(v_{q,0})=\tau\Big(\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}(v_{q+1,\ell(\sigma)})\Big)=\tau\Big(\mathcal{I}\big(q+1,\ell(\sigma)\big)\Big). \tag{3.6}$$

By noting $\tau\sigma=t_{\varpi_m}$, we see that

$$\tau(e_{\sigma(\alpha)}\otimes t^s)=e_\alpha\otimes t^{s-\langle h_\alpha,\varpi_m\rangle}\ \text{ for } \alpha\in\Delta, s\in\mathbb{Z}_{\geq 0}. \tag{3.7}$$

Lemma 3.4. *For $\alpha\in\Delta[-1]$ and $r,s,k\in\mathbb{Z}_{>0}$ we have*

$$\tau\big({}_k e_{\sigma(\alpha)}(r,s)\big)={}_{k+1}e_\alpha(r,s+r).$$

Proof. It is easily seen that the map

$${}_k\mathbf{S}(r,s)\ni(b_j)_{j\geq 0}\to(b'_j)_{j\geq 0}\in{}_{k+1}\mathbf{S}(r,s+r)$$

defined by $b'_j=b_{j-1}$ is bijective. Then the assertion is proved from (3.7). □

Using (3.7) and the above lemma, we see that

$$\begin{aligned}\tau\Big(\mathcal{I}\big(q+1,\ell(\sigma)\big)\Big)=\sum_{\alpha\in\Delta[1]}U(\widehat{\mathfrak{n}}_+)\Big\{\mathbb{C}(f_\alpha\otimes t)^{L_{q+1}+1} \\ +\sum_{(r,s,k)\in T_{q+1}}\mathbb{C}\,{}_{k+1}f_\alpha(r,s+r)\Big\}+U(\widehat{\mathfrak{n}}_+)\big(\mathfrak{n}_++t\mathfrak{p}_{\varpi_m}[t]\big).\end{aligned} \tag{3.8}$$

On the other hand, we have

$$\mathcal{I}(q,0)=\sum_{\alpha\in\Delta[1]}\sum_{(r,s,k)\in T_q}U(\widehat{\mathfrak{n}}_+)_kf_\alpha(r,s)+U(\widehat{\mathfrak{n}}_+)\big(\mathfrak{n}_++t\mathfrak{p}_{\varpi_m}[t]\big). \tag{3.9}$$

It is easily checked that

$$\big\{(r,s+r,k+1)\ \big|\ (r,s,k)\in T_{q+1}\big\}=\{(r,s,k)\in T_q\mid k>1\},$$

which implies

$$\sum_{(r,s,k)\in T_{q+1}}U(\widehat{\mathfrak{n}}_+)_{k+1}f_\alpha(r,s+r)=\sum_{\substack{(r,s,k)\in T_q\\ k>1}}U(\widehat{\mathfrak{n}}_+)_kf_\alpha(r,s). \tag{3.10}$$

We shall prove for each $\alpha\in\Delta[1]$ that

$$(f_\alpha\otimes t)^{L_{q+1}+1}\in\sum_{(r,s,k)\in T_q}U(\widehat{\mathfrak{n}}_+)_kf_\alpha(r,s),\quad\text{and} \tag{3.11}$$

$${}_1f_\alpha(r,s)\in U(\widehat{\mathfrak{n}}_+)\Big(\mathbb{C}(f_\alpha\otimes t)^{L_{q+1}+1}+\mathfrak{n}_+[t]+t\mathfrak{h}[t]\Big)\ \text{ for } r,s \text{ with } (r,s,1)\in T_q. \tag{3.12}$$

By comparing (3.8) and (3.9) and using (3.10), we see that these imply

$$\tau\Big(\mathcal{I}\big(q+1,\ell(\sigma)\big)\Big) = \mathcal{I}(q,0),$$

and then by (3.6) we have $\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)}(v_{q,0}) = \mathcal{I}(q,0)$, which completes the proof. Setting $r = s = L_{q+1}+1$, we have

$${}_1f_\alpha(r,s) = (f_\alpha \otimes t)^{L_{q+1}+1},$$

and hence (3.11) follows. Assume that r, s satisfy $(r,s,1) \in T_q$, which implies $s \geq L_{q+1}+1$. Since ${}_1f_\alpha(r,s) = 0$ if $r > s$, we may assume $r \leq s$. If we apply the Lie algebra automorphism $\tau \circ \Phi_\sigma$ to (2.2) with s replaced by $s-r$, we have from Lemma 3.4 that

$$e_\alpha^{(s-r)}(f_\alpha \otimes t)^{(s)} - (-1)^{s-r}\,{}_1f_\alpha(r,s) \in U(\widehat{\mathfrak{n}}_+)\big(\mathfrak{n}_+[t] \oplus t\mathfrak{h}[t]\big).$$

Hence (3.12) also holds.

Acknowledgement

The author is supported by JSPS Grant-in-Aid for Young Scientists (B) No. 25800006.

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Katsuyuki Naoi, Institute of Engineering, Tokyo University of Agriculture and Technology, 2-24-16 Naka-cho, Koganei-shi, Tokyo 184-8588, JAPAN, e-mail: naoik@cc.tuat.ac.jp